SOME DEGENERATE TRANSONIC FLOWS

(NEKOTORYE VYROZHDENNYYE OKOLOZVUKOVYYE TECHENIIA)

PMM Vol.22, No.2, 1958, pp.260-264

O.S. RYZOV
(Moscow)

(Received 9 June 1957)

We consider transonic motions of an ideal gas which are represented in the velocity hodograph by a curve or by a surface. In the second part of the note we determine a class of self-similar solutions representing plane and axially symmetric flows.

1. Three-dimensional flows with a degenerate hodograph. 1. The equations of a transonic three-dimensional gas flow in cartesian coordinates read

$$uu_x - v_y - w_z = 0, \quad u_y - v_x = 0, \quad u_z - w_x = 0, \quad v_z - w_y = 0$$
$$u = (\varkappa + 1)\frac{U}{a_*}, \qquad v = (\varkappa + 1)\frac{V}{a_*}, \qquad w = (\varkappa + 1)\frac{W}{a_*} \tag{1.1}$$

Here κ is the adiabatic exponent, U, V, W the perturbation velocity components along the x, y, z axes, the undisturbed velocity vector having the magnitude of the critical speed a_* and being directed along the x axis.

Let us consider double waves, that is flows for which only the two quantities v and w are independent, and

$$u = u(v, w) \tag{1.2}$$

Using (1.2) and equations (1.1) we obtain

$$(uu_v^2 - 1) v_y + 2uu_v u_w v_z + (uu_w^2 - 1) w_z = 0 \tag{1.3}$$

Every plane x = const. of the physical space is mapped in the velocity hodograph space onto the same surface Σ. Hence we may consider in equations (1.3) the variables v and w as independent, and the variables y and z as functions of these independent variables. We have

$$v_y = z_w \Delta, \qquad v_z = -y_w \Delta, \qquad w_z = y_v \Delta, \qquad \Delta = v_y w_z - v_z^2$$

and by equation (1.3)

$$(uu_v^2 - 1) z_w - 2uu_v u_w y_w + (uu_w^2 - 1) y_v = 0 \tag{1.4}$$

Consider now the function χ determined by the equation [1]

Reprint Order No. PMM 31.

$$\chi = ux + vy + wz - \varphi \qquad (u = \varphi_x,\ v = \varphi_y,\ w = \varphi_z) \tag{1.5}$$

Its differential is $d\chi = (y + xu_v)dv + (z + xu_w)dw$. Hence

$$\chi_v = y + xu_v, \qquad \chi_w = z + xu_w \tag{1.6}$$

Differentiating equations (1.6) for x = const. we obtain

$$y_v = \chi_{vv} - xu_{vv}, \qquad y_w = \chi_{vw} - xu_{vw}, \qquad z_w = \chi_{ww} - xu_{ww} \tag{1.7}$$

Now we substitute relations (1.7) into equation (1.4). Setting the term not containing z and the term containing z in the first degree equal to zero, we obtain two equations which determine the functions w and χ :

$$(uu_w^2 - 1)\, u_{vv} - 2uu_v u_w u_{vw} + (uu_v^2 - 1)\, u_{ww} = 0$$
$$(uu_w^2 - 1)\, \chi_{vv} - 2uu_v u_w \chi_{vw} + (uu_v^2 - 1)\, \chi_{ww} = 0 \tag{1.8}$$

After equations (1.8) are solved, formulas (1.6) [1] give the transformation to the physical space. In the case of conical flows we have $\chi = 0$ and $y/x = -u_v$, $z/x = -u_w$.

2. We note one particular solution of the first equation (1.8).

One easily sees that this equation admits the transformation group $U(v, w) = a_1^{-2/3} u(a_1 v, a_1 w)$, where a_1 is an arbitrary non-vanishing constant. Therefore equations (1.8) have a solution of the form

$$u = v^{2/3} f_1(\xi_1), \qquad \xi_1 = w / v \tag{1.9}$$

where the function f_1 satisfies the ordinary differential equation

$$\left(\xi_1^2 - 2\xi_1^2 f_1 f_1'^2 + \frac{4}{9} f_1^3\right) f_1'' + \frac{2}{9} f_1^2 f_1'^2 - \frac{2}{3}\xi_1 f_1' + \frac{2}{9} f_1 = 0 \tag{1.10}$$

This equation is itself invariant under the transformation group $\Phi_1(\xi_1) = a_2^{-2/3} f(a_2 \xi_1)$, therefore its order may be lowered. Indeed, setting $f_1 = \xi_1^{2/3} F_1(\eta_1)\eta_1 = \ln|\xi_1|$, we transform equation (1.10) into an equation which does not contain the independent variable explicitly. Next, setting $dF_1/d\eta_1 = \Psi_1$ and taking F_1 as the independent variable, we obtain the first order equation

$$\frac{d\Psi_1}{dF_1} = \frac{{}^4/_9 F_1 + {}^1/_3 \Psi_1 + {}^2/_3 F_1 \Psi_1^3 + {}^2/_9 F_1^2 \Psi_1^2 - {}^4/_9 F_1^3 \Psi_1 - {}^{16}/_{81} F_1^4}{\Psi_1 (1 - {}^4/_9 F_1^3 - {}^8/_3 F_1^2 \Psi_1 - 2F_1 \Psi_1^2)} \tag{1.11}$$

Solving equation (1.11) and carrying out the second integration

$$\eta_1 - \eta_0 = \int \frac{dF_1}{\Psi_1(F_1)}$$

where η_0 is an arbitrary constant, we go over to the physical space by the formulas

$$\frac{y}{x} = \frac{w^{2/3}}{v} \frac{dF_1}{d\eta_1}, \qquad \frac{z}{x} = \frac{1}{w^{1/3}} \left(\frac{2}{3} F_1 + \frac{dF_1}{d\eta_1}\right) \tag{1.12}$$

3. Let us consider axially symmetric flows of the class discussed above. We have that

$$u = u(\omega), \qquad \omega = \sqrt{v^2 + w^2} \tag{1.13}$$

and the first equation (1.8) implies that

$$\omega u'' - uu'^3 + u' = 0 \tag{1.14}$$

Interchanging dependent and independent we obtain the well known equation solved by Busemann [2]:

$$\omega\omega'' = \omega'^2 - u \tag{1.15}$$

4. The differential equation of the characteristics of the first equation (1.8) reads

$$(uu_w^2 - 1)\,dw^2 + 2uu_v u_w dv dw + (uu_v^2 - 1)\,dv^2 = 0 \tag{1.16}$$

Noting that along the surface Σ we have $du = u_v dv + u_w dw$, we obtain from (1.16) the equation

$$u du^2 = dv^2 + dw^2 \tag{1.17}$$

This equation determines characteristic curves S_+ and S_- on the surface Σ, One can show that simple waves are described by such an equation. In this case the whole flow is mapped onto a single S curve in the velocity hodograph.

The formulas

$$u = \left(\tfrac{3}{2}\tau\right)^{2/3}, \qquad v = \int \cos f_2(\tau)\, d\tau, \qquad w = \int \sin f_2(\tau)\, d\tau \tag{1.18}$$

solve equation (1.17). The corresponding flow in the physical space is given by the relation

$$({}^3/_2\,\tau)^{-1/3} x + \cos f_2(\tau)\, y + \sin f_2(\tau)\, z + F_2(\tau) = 0 \tag{1.19}$$

where $f_2(\tau)$ and $F_2(\tau)$ are arbitrary functions of τ. This solution can be used in computing certain aerofoils.

2. Self-similar plane and axially symmetric flows. 1. We consider now plane and axially symmetric flows. In this case we have by (1.1)

$$-uu_x + \omega_r + \delta\omega/r = 0, \qquad u_r = \omega_x \tag{2.1}$$

Here ω is the component of the perturbation velocity (1) in the direction of the radius r, $\delta = 0$ for plane flows and $\delta = 1$ for axially symmetric flows.

The system of equations (2.1) is invariant under the continuous transformation group

$$U(x, r) = \alpha_3^{2(1-\beta)} u(\alpha_3^\beta x, \alpha_3 r), \qquad \Omega(x, r) = \alpha_3^{3(1-\beta)} \omega(\alpha_3^\beta x, \alpha_3 r) \tag{2.2}$$

where α_3 and β are arbitrary constants. Hence system (2.1)must possess self-similar solutions of the form [3, 7]

$$u = x^{2(\beta+1)} f_3(\xi_2), \qquad \omega = x^{3(\beta+1)} f_4(\xi_2), \qquad \xi_2 = r x^\beta \tag{2.3}$$

In fact, substituting the expressions (2.3) into the equations (2.1) we obtain

$$f_3' = 3(\beta+1) f_4 + \beta\xi_2 f_4', \qquad -2(\beta+1) f_3^2 - \beta\xi_2 f_3 f_3' + f_4' + \delta f_4 / \xi_2 = 0 \quad (2.4)$$

Eliminating the function f_4 from the system (2.4) we obtain a single second order equation for the function f_3 :

$$(\beta^2\xi_2^2 f_3 - 1) f_3'' + (9\beta + 7)\beta\xi_2 f_3 f_3' + \beta^2\xi_2^2 f_3'^2 + 2(4\beta^2 + 7\beta + 3) f_3^2 - \delta f_3' / \xi_2 = 0 \quad (2.5)$$

It is easy to see that equation (2.5) is itself invariant under the transformation group $\Phi_3(\xi_2) = a_4^2 f_3(a_4\xi_2)$, where a_4 is an arbitrary constant. Setting $f_3 = \xi_2^{-2} F_3(\eta_2)$, $\eta_2 = \ln|\xi_2|$ we transform equation (2.5) into an equation which does not contain the independent variable explicity. Next, setting $f_3 = \xi_2^{-2} F_3(\eta_2)$, $\eta_2 = \ln|\xi_2|$, and taking F_2 as the independent variable we obtain the first order equation

$$\frac{d\Psi_3}{dF_3} = \frac{(\delta-5)\Psi_3 + 2(3-\delta)F_3 - \beta^2\Psi_3^2 - 7\beta\Psi_3 F_3 - 6F_3^2}{\Psi_3(\beta^2 F_3 - 1)} \quad (2.6)$$

The functions f_4 and F_3 are connected by the relation

$$f_4 = \frac{\xi_2^{-3}}{3\beta + 3 - \delta\beta}(F_3' - 2F_3 - 2\beta F_3^2 - \beta^2 F_3 F_3') = \xi_2^{-3} R(\eta_2) \quad (2.7)$$

The perturbation velocity components are given by

$$u = \left(\frac{x}{r}\right)^2 F_3(\eta_2), \qquad \omega = \left(\frac{x}{r}\right)^3 R(\eta_2) \quad (2.8)$$

2. Consider the case $\beta = 0$. Then $u = x^2 f_3(r)$, $\omega = x^3 f_4(r)$, so that the variables x and r are separated and we obtain solutions studied by Zigulev [4]

$$f_3 = \zeta_2^{-2(\beta+1)} f_5(\zeta_2), \qquad f_4 = \zeta_2^{-3(\beta+1)} f_6(\zeta_2), \qquad \zeta_2 = \xi_2^{1/\beta}, \quad 1/\beta = \gamma \quad (2.9)$$

Using (2.4) we obtain for the functions $f_5(\zeta_2)$ and $f_6(\zeta_2)$ the system of equations

$$-2(\gamma+1) f_5 + \gamma\zeta_2 f_5' = f_6', \qquad -f_5 f_5' + (\delta - 3 - 3\gamma) f_6 + \gamma\zeta_2 f_6' = 0 \quad (2.10)$$

Consider, in particular, the case $\gamma = 0$. Then $u = r^{-2} f_5(x)$, $\omega = r^{-3} f_6(x)$, that is, the variables x and r are separated and we obtain another type of flows studied in the paper just quoted.

3. Consider now the case $\beta = -½$. It is easy to verify that in this case equation (2.6) admits the solution

$$\Psi_3 = 2(1+\delta)\left(1 + \frac{2}{1+\delta}F_3 \pm \sqrt{1 + \frac{2}{1+\delta}F_3}\right) \quad (2.11)$$

Using (2.11) one easily obtains the relation

$$f_3 = A + \frac{A^2}{2(1+\delta)}\xi_2^2 \quad (2.12)$$

The perturbation velocity components are now given by

$$u = Ax + \frac{A^2}{2(1+\delta)}r^2, \qquad \omega = \frac{A^2}{1+\delta}xr + \frac{A^3}{2(1+\delta)(3+\delta)}r^3 \quad (2.13)$$

If the gas flow considered is two-dimensional, so that $\delta = 0$, then the relations (2.13) are the solution due to Falkovich [5] which describe the shockless flow in a nozzle near the sonic line. If $\delta = 1$, the equations (2.13) describe an axially symmetric flow in an axially symmetric nozzle. The equation of the sonic line reads

$$x = -\frac{A}{2(1+\delta)} r^2 \tag{2.14}$$

This equation shows that in an axially symmetric nozzle the sonic line is closer to a vertical straight line than it would be in a plane nozzle for the same value of the constant A. This constant equals the derivative u_x at the center of the nozzle.

3. Plane and axially symmetric flows limiting to self-similar ones. 1. We consider first plane transonic flows. In this case $\delta = 0$ and equations (2.1) are invariant under the transformations $r = r_0 + r'$. We use the method of the paper [6] in which it was shown how to obtain, by a limiting process, a new class of solutions from a class of self-similar solutions containing an arbitrary exponent which may be increased indefinitely. The solutions given in 2(2) have this property. After some calculations we obtain

$$u = e^{-2mr} f_7(\xi_3), \qquad \omega = e^{-3mr} f_8(\xi_3), \qquad \xi_3 = xe^{mr} \tag{3.1}$$

where the functions f_7 and f_8 satisfy the system of ordinary differential equations

$$-f_7 f_7' - 3mf_8 + m\xi_3 f_8' = 0, \qquad -2mf_7 + m\xi_3 f_7' = f_8' \tag{3.2}$$

(a limiting form of the equations in 2(2)).

Eliminating from (3.2) the function f_8 we obtain a single second order differential equation for the function f_7:

$$(m^2\xi_3^2 - f_7) f_7'' - f_7'^2 - 3m'\xi_3 f_7' + 4m^2 f_7 = 0 \tag{3.3}$$

This equation is invariant under the group $\Phi_4(\xi_3) = a_5^{-2} f_7(a_5\xi_3)$ and can be reduced to a first order equation

$$\frac{d\Psi_4}{dF_4} = \frac{6F_4^2 + 7F_4\Psi_4 + \Psi_4^2}{\Psi_4(m^2 - F_4)} \tag{3.4}$$

Here

$$\eta_3 = \ln|\xi_3|, \qquad f_7 = \xi_3^2 F_4(\eta_3), \qquad \Psi_4 = \frac{dF_4}{d\eta_3} \tag{3.5}$$

The perturbation velocity components are given by

$$u = x^2 F_4$$
$$\omega = \frac{x^3}{3m}(m^2 F_4' - F_4 F_4' - 2F_4^2) \tag{3.6}$$

If $m = 0$, then the functions f_7 and f_8 are constnat and we have a uniform flow. For $m \neq 0$ the integral curves of equation (3.4) are shown on Fig. 1. The essential characteristic of the flows just considered is their asymmetry with respect to the axis $r = 0$.

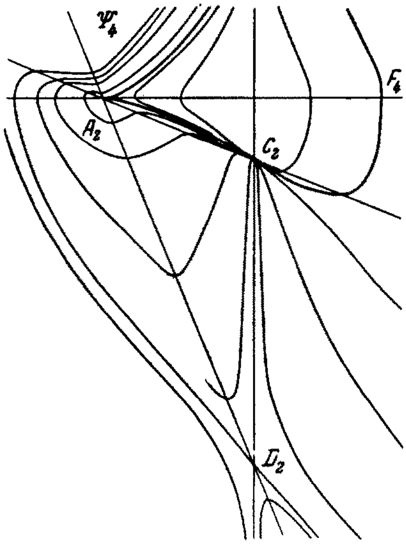

Fig. 1.

2. We consider now plane and axially symmetric flows simultaneously. Using the invariance of equations (2.1) with respect to the transformations $x = x_0 + x'$ we obtain from relations (2.3) a new class of solutions. These solutions, obtained by letting the exponent β increase indefinitely, have the form [6]

$$u = e^{2nx} f_9(\xi_4), \qquad \omega = e^{3nx} f_{10}(\xi_4), \qquad \xi_4 = re^{nx} \tag{3.7}$$

where the functions f_9 and f_{10} are determined by the equations

$$-2nf_9^2 - n\xi_4 f_9 f_9' + f_{10}' + \delta f_{10} / \xi_4 = 0, \qquad f_9' = 3nf_{10} + n\xi_4 f_{10}' \tag{3.8}$$

(limiting case of equations (2.3)). The function f_9 must satisfy the equation

$$(n^2\xi_4^2 f_9 - 1) f_9'' + 9n^2\xi_4 f_9 f_9' + n^2\xi_4^2 f_9'^2 + 8n^2 f_9^2 - \delta f_9' / \xi_4 = 0 \tag{3.9}$$

As before we introduce new variables

$$\eta_4 = \ln \xi_4, \quad f_9 = \xi_4^{-2} F_5(\eta_4), \qquad \Psi_5 = \frac{dF_5}{d\eta_4} \tag{3.10}$$

and obtain a single first order equation

$$\frac{d\Psi_5}{dF_5} = \frac{(\delta - 5)\Psi_5 + 2(3 - \delta)F_5 - n^2\Psi_5^2}{\Psi_5(n^2F_5 - 1)}$$

BIBLIOGRAPHY

1. Nikol'skii, A.A., O klasse adiabaticheskikh techenii gaza, kotorye v prostranstve godografa skorosti izobrazhazhaintsya poverkhnostiami (On a class of adiabatic gas streams which express themselves as surfaces in the range of the speed hodograph). *Trudy. tsent. aviats. gos. univ.*, 1949.

2. Busemann, A., Die achsesymmetrische Kegelige Uberschallstromung. *Luftfahrtforschung*, Vol. 19, 1942.

3. Morgan, A.J.A., On a class of two-dimensional channel flows with a straight sonic line. *Journ. Aer. Sci.* Vol. 22, No. 8, 1955.

4. Zhigulev, V.N., Ob odnom klasse ploskikh i osesimmetricheskikh okolozvukovykh techenii (On a class of axiosymmetrical flows at speeds near the speed of sound). *PMM* Vol. 20, No. 5, 1956.

5. Fal'kovich, S.V., K teorii sopla Lavalia (On Lavalle's theory of the nozzle). *PMM* Vol. 10, No. 4, 1956.

6. Barenblatt, G.I., O predel'nykh avtomodel'nykh dvizheniiakh v teorii nestatsionarnoi fil'tratsii gaza v poristoi srede i teorii pogranichnogo sloia (Limited self-manifesting movements in the theory of the unsteady filtration of a gas in a porous medium and boundary layer theory). *PMM* Vol. 18, No. 4, 1954.

7. Guderley, G. and Joshihara, H., An axial-symmetric transonic flow pattern. *Quart. appl. math.* Vol. 8, No. 4, 1951.

Translated by L.B.